Some addition formulae for Abelian functions for elliptic and hyperelliptic curves of cyclotomic type

J. C. Eilbeck[1], S. Matsutani[2], Y. Ônishi[3]
[1]Department of Mathematics and
the Maxwell Institute for Mathematical Sciences,
Heriot-Watt University, Edinburgh EH14 4AS, UK
[2] 8-21-1 Higashi-Linkan,
Sagamihara, 228-0811, Japan.
[3] 4-3-11, Takeda, Kofu, 400-8511
Faculty of Education and Human Sciences
University of Yamanashi, Japan.

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Abstract

We discuss a family of multi-term addition formulae for Weierstrass functions on specialized curves of low genus with many automorphisms, concentrating mostly on the case of genus one and two. In the genus one case we give addition formulae for the equianharmonic and lemniscate cases, and in genus two we find some new addition formulae for a number of curves.

1 Introduction

The aim of this paper is to introduce some new addition formulae for the Weierstrass σ and $\wp$ functions in genus one, and some generalisations to some higher genus cases. These formula are found in the special case when some of the coefficients (moduli) of the associated algebraic curves are chosen to be zero, and as a result the curves have additional automorphisms (extra symmetries).

Although elliptic functions, including the Weierstrass elliptic functions, have been extensively used (or perhaps over-used) to enumerate travelling wave solutions of nonlinear wave equations, relatively little has been written about the correspondingly higher genus generalisations. This is partly because no general handbooks exist which play the same role as the familiar treatises on elliptic functions. This paper is part of a project to provide the material for such a compendium.

Those coming to this paper because of possible applications to number theory may prefer to see it as extending the classical theory of complex multiplication for elliptic functions to higher genus functions. In this paper the complex multiplications are of "cyclotomic type", i.e. involving complex roots of unity. These generalise the more well-known addition formulae involving results for $f(u+v)f(u-v)$ where we think of the $(\pm 1)v$ as involving the two real roots of unity.

We begin by summarising well-documented existing results for the genus one case. In this case we start with an elliptic curve reduced to the standard Weierstrass form

$$y^2 = 4x^3 - g_2 x - g_3. \tag{1.1}$$

The function $\wp(u)$ is the inverse function $u \mapsto x$ determined by

$$u = \int_{\infty}^{(x,y)} \frac{dx}{2y}, \tag{1.2}$$

and $\sigma(u)$ is an entire function satisfying

$$\wp(u) = -\frac{d^2}{du^2} \log \sigma(u). \tag{1.3}$$

The function $\wp(u)$ satisfies the well-known formula

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{1.4}$$

which is isomorphic to the genus one curve (but note this result does not hold for the higher genus cases).

The following two-variable addition formula plays an important role in the theory of the Weierstrass σ and $\wp$ functions, and its generalisation are central to this paper:

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(u) - \wp(v). \tag{1.5}$$

Taking the second logarithmic derivative of (1.5) gives the well known addition formula involving just $\wp$ and $\wp'$, which is also an addition formula on the curve (1.1).

A three-variable addition formula is also known from the work of Frobenius and Stickelburger [11] (see also Whittaker & Watson, [17])

$$\frac{\sigma(u-w)\sigma(v-w)\sigma(u-v)\sigma(u+v+w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3} = -\frac{1}{2}\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w) \end{vmatrix}. \tag{1.6}$$

In the genus two case, starting with the hyperelliptic curve

$$y^2 = x^5 + \mu_2 x^4 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10}, \tag{1.7}$$

one can define generalized σ and $\wp$ functions (see the classical book by Baker [3], or Buchstaber *et al.* [5] for a modern treatment). The main difference is that

σ is now a function of $g = 2$ variables, $u = \{u_1, u_2\}$, and there are now three possible versions of the $\wp$ function, due to the different possible logarithmic differentials of the σ function:

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad 1 \le i \le j \le 2. \tag{1.8}$$

(Note that in this notation the $\wp$ of the genus one theory would be written as $\wp_{11}$). The functions $\wp_{ij}$ and $\wp_{ijk} = \frac{\partial}{\partial u_k}\wp_{ij}(u)$ satisfy equations analogous to (1.4) The genus two σ and $\wp$ functions satisfy an analogue of the elliptic addition formula (1.5)

$$-\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u)\wp_{22}(v) - \wp_{22}(u)\wp_{12}(v) \tag{1.9}$$

as described in Baker [2, 3]. A generalisation of (1.6) in the genus two case has also been derived (Eilbeck *et al.* [10]), but this is a rather complicated formula.

Our main aim in this paper is to point out that Abelian functions associated with a curve with many automorphisms, namely, with extra symmetries relative to general case, have novel addition formulae, which are *not* valid in the general case. In addition, we wish to present the addition formulae in a way which makes the extra symmetries explicit.

Although we restrict ourselves mostly to elliptic ($g = 1$) and hyperelliptic curves ($g = 2$) in this paper, we comment briefly on similar results which have been derived or are under study for more general curves.

As an example, whilst the only nontrivial automorphism of the curve (1.1) with generic values of g_js is $(x, y) \mapsto (x, -y)$, the special curve

$$y^2 = x^3 - g_3 \quad \text{with } g_3 \neq 0$$

has six automorphisms $(x, y) \mapsto (\zeta^j x, \pm y)$ with $j = 0, 1, 2$ and $\zeta = \exp(2\pi i/3)$. This has other addition formulae different from (1.5). We mention here one from Theorem 6.4:

$$-\frac{\sigma(u \pm v)\sigma(u \pm \zeta v)\sigma(u \pm \zeta^2 v)}{\sigma(u)^3\sigma(v)^3} = \pm\frac{1}{2}\left(\wp'(u) \pm \wp'(v)\right).$$

The formula is novel in the sense that, although it can be derived from (1.5), it is only valid in case that $g_2 = 0$.

As another example, while the curve (1.7) with generic parameters has only two automorphisms $(x, y) \mapsto (x, \pm y)$, the special curve

$$y^2 = x^5 + \mu_{10}, \quad \text{with } \mu_{10} \neq 0, \tag{1.10}$$

has ten automorphisms, $(x, y) \mapsto (\zeta^i x, \pm y)$, $j = 0, 1, \ldots, 4$, with $\zeta = \exp(2\pi i/5)$, and Abelian functions on its Jacobian variety has addition formula different from (1.9), for example Proposition 7.3 that expresses

$$\frac{\sigma(u+v)\sigma(u+[\zeta]v)\sigma(u+[\zeta^2]v)\sigma(u+[\zeta^3]v)\sigma(u+[\zeta^4]v)}{\sigma(u)^5\sigma(v)^5}, \tag{1.11}$$

as a polynomial in $\wp_{ij}(u)$, $\wp_{ij}(v)$, $\wp_{ijk}(u)$, and $\wp_{ijk}(v)$. Here $\zeta = \exp(2\pi i/5)$ and $[\zeta^j]v = [\zeta^j](v_1, v_2) = (\zeta^j v_1, \zeta^{2j} v_2)$.

In genus one case, we obtain also three-term and four-term addition formulae by using (1.6). While formulae of a similar type exist in the higher genus cases, we do not mention them here because of their complexity.

We have two ways to prove these formulae. One is by simplifying the expression in terms of $\wp$ and its derivative given by taking product of modified formulae from (1.5) or (1.6), or their higher genus generalizations. The other is similar to the method used in [8], that is by balancing an expression such as (1.11) with a linear combination of suitable $\wp$-functions, in which the derivation of the correct coefficients are aided by algebraic computing software.

The paper is laid out as follows. We first cover some basic theory mainly needed for genus two and higher. In §2, we review the types of curves we consider in the paper. After introducing basic notions in §3, we define the function $\sigma(u)$ in §4, and the $\wp$-functions in §5. We consider the genus one case in §6, giving some detail to provide a pedagogical background for the general methods. The genus two case is discussed in §7. In §8 we discuss briefly further generalisations to higher genus cases, a topic which will be covered in more detail elsewhere.

2 Elliptic and hyperelliptic curves of cyclotomic type

In this section, we describe clearly the curves which we shall consider. Let $a \geq 2$ and m be positive integers. We consider two type of curves according to am is odd or even. Namely, let

$$f(x,y) = \begin{cases} y^2 + \mu_{am}\, y - (x^{am} + \mu_{2a}x^{a(m-1)} + \mu_{4a}x^{a(m-2)} + \cdots \\ \qquad\qquad + \mu_{2a(m-1)}x^{a} + \mu_{2am}), & \text{if } am \text{ is odd,} \\ y^2 - (x^{am+1} + \mu_{2a}x^{a(m-1)+1} + \mu_{4a}x^{a(m-2)+1} + \cdots \\ \qquad\qquad + \mu_{2a(m-1)}x^{a+1} + \mu_{2am}x), & \text{if } am \text{ is even.} \end{cases} \tag{2.1}$$

We consider the projective curve C defined by the affine equation

$$f(x,y) = 0$$

with adding the unique point ∞ at infinity. The genus of C is

$$g = \lfloor am/2 \rfloor$$

if it is non-singular. We refer the curve that is defined by the former equation as the $(2, a[m])$-curve, and the later as the $(2, a[m]{+}1)$-curve. Here the first entry the number "2" indicates these curves are either elliptic or hyperelliptic curves, namely, the power of y in the defining equation. We are aiming to treat any algebraic curves with a unique point at infinity, but we restrict ourselves here to elliptic and hyperelliptic curves and to present our idea simply.

Both the $(2, a[m])$-curve and the $(2, a[m]{+}1)$-curve are acted on by the group W_{2a} of $2a$-th roots of 1 as automorphisms:

$$\begin{aligned} [-\zeta^2] : (x, y) \mapsto & (\zeta^2 x, -y - \mu_{am}) && \text{for a } (2, a[m])\text{-curve}, \\ [-\zeta] : (x, y) \mapsto & (-\zeta x, \boldsymbol{i} y) && \text{for a } (2, a[m]{+}1)\text{-curve}, \end{aligned} \tag{2.2}$$

where ζ is an $2a$-th root of 1, and $\boldsymbol{i}^2 = -1$.
Examples: We give some examples here:

1. The general $(2, 3[1])$-curve is defined by $y^2 + \mu_3 y = x^3 + \mu_6$. This is acted on by W_6.
2. The general $(2, 2[1]{+}1)$-curve is defined by $y^2 = x^3 + \mu_4 x$. This is acted on by W_4.
3. The general $(2, 2[2]{+}1)$-curve is defined by $y^2 = x^5 + \mu_4 x^3 + \mu_8 x$, which is the famous Burnside curve. This is acted on by W_4.
4. The general $(2, 5[1])$-curve is defined by $y^2 + \mu_5 y = x^5 + \mu_{10}$. This is acted on by W_{10}.
5. The general $(2, 4[1]{+}1)$-curve is defined by $y^2 = x^5 + \mu_8 x$. This is acted on by W_8.
6. The general $(2, 3[2]{+}1)$-curve is defined by $y^2 = x^7 + \mu_6 x^4 + \mu_{12} x$. This is acted on by W_6.
7. The general $(2, 2[3]{+}1)$-curve is defined by $y^2 = x^7 + \mu_4 x^5 + \mu_8 x^3 + \mu_{12} x$. This is acted on by W_4.
8. The general $(2, 7[1])$-curve is defined by $y^2 + \mu_7 y = x^7 + \mu_{12} x$. This is acted on by W_{14}.

In this paper we suppose that μ_j is 0, if it does not appear in the equation of C, $f(x, y) = 0$.

3 Differential forms, etc.

In the hyperelliptic case, the space of differential forms are spanned by

$$\omega_1 = \frac{dx}{2y}, \quad \omega_2 = \frac{x dx}{2y}, \quad \cdots, \quad \omega_g = \frac{x^{g-1} dx}{2y}, \tag{3.1}$$

For variable g points (x_1, y_1), (x_2, y_2), $\cdots$, (x_g, y_g) on C, we consider the integrals

$$\begin{aligned} u &= (u_1, u_2, \cdots, u_g) \\ &= \int_\infty^{(x_1,y_1)} \omega + \int_\infty^{(x_2,y_2)} \omega + \cdots + \int_\infty^{(x_g,y_g)} \omega, \end{aligned} \tag{3.2}$$

where

$$\omega = (\omega_1, \omega_2, \cdots, \omega_g). \tag{3.3}$$

Let

$$\eta_j = \frac{1}{2y} \sum_{k=j}^{2g-j} (k+1-j)\mu_{4g-2k-2j} x^k dx \quad (j = 1, \cdots, g), \tag{3.4}$$

which are differential forms of the second kind without poles except at ∞.

Let Λ be the lattice in $\mathbb{C}^g$ generated by the loop integrals of ω:

$$\Lambda = \left\{ \oint \omega \right\}. \tag{3.5}$$

Then the Jacobian variety of C is given by $\mathbb{C}^g/\Lambda$. For $k = 1, 2, \cdots, g$, the map

$$\begin{array}{rcl} \iota : \mathrm{Sym}^k(C) & \to & J, \\ (P_1, \cdots, P_k) & \mapsto & \left(\displaystyle\int_\infty^{P_1} \omega + \cdots + \int_\infty^{P_k} \omega \right) \bmod \Lambda, \end{array} \tag{3.6}$$

is an injection outside a certain small dimensional (relative to k) subset. If $k = g$, the map is surjective. We denote the image $\iota(\mathrm{Sym}^k(C))$ by $\Theta^{[k]}$. Let

$$\begin{array}{rcl} R_1 & = & \mathrm{rslt}_x\big(\mathrm{rslt}_y\big(f(x,y), f_x(x,y)\big), \mathrm{rslt}_y\big(f(x,y), f_y(x,y)\big)\big), \\ R_2 & = & \mathrm{rslt}_y\big(\mathrm{rslt}_x\big(f(x,y), f_x(x,y)\big), \mathrm{rslt}_x\big(f(x,y), f_y(x,y)\big)\big), \\ R_3 & = & \gcd(R_1, R_2), \end{array} \tag{3.7}$$

where rslt_z represents the resultant, namely, the determinant of the Sylvester matrix with respect to the variable z. Then R_3 is a perfect square in the ring

$$\mathbb{Z}[\{\mu_j\}].$$

Hence we define

$$D = {R_3}^{1/2}. \tag{3.8}$$

4 The sigma function

4.1 The definition of $\sigma(u)$

We define here an entire function $\sigma(u) = \sigma(u_1, \cdots, u_g)$ on $\mathbb{C}^g$ associated with C, which we call the *σ-function*. As usual, let

$$\alpha_i, \ \beta_j \quad (1 \leq i, \ j \leq g) \tag{4.1}$$

be closed paths on C which generate $H_1(C, \mathbb{Z})$ such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$, $\alpha_i \cdot \beta_j = \delta_{ij}$.

Define the period matrices by

$$[\omega' \ \omega''] = \left[\int_{\alpha_i} \omega_j \quad \int_{\beta_i} \omega_j\right]_{i,j=1,\cdots,g}, \ [\eta' \ \eta''] = \left[\int_{\alpha_i} \eta_j \quad \int_{\beta_i} \eta_j\right]_{i,j=1,\cdots,g}. \tag{4.2}$$

From (3.1) we see the canonical divisor class of C is given by 4∞, and we are taking ∞ as the base point of the Abel map (3.6) for $k = g$. Hence the Riemann constant is an element of $\left(\frac{1}{2}\mathbb{Z}\right)^{2g}$ (see Mumford [13]), Coroll.3.11, p.166). Let

$$\delta = \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in \left(\tfrac{1}{2}\mathbb{Z}\right)^{2g} \tag{4.3}$$

be the theta characteristic which gives the Riemann constant with respect to the base point ∞ and the period matrix $[\omega' \ \omega'']$. Note that we use δ', δ'' as well as n in (4.4) as columns, to keep the notation a bit simpler. We define

$$\begin{aligned}
\sigma(u) &= \sigma(u_1, \cdots, u_g) \\
&= c \exp(-\frac{1}{2} u\eta'{\omega'}^{-1}\,{}^t u)\Theta[\delta]\,({\omega'}^{-1}\ {}^t u;\ {\omega'}^{-1}\omega'') \\
&= c \exp(-\frac{1}{2} u\eta'{\omega'}^{-1}\ {}^t u) \sum_{n\in\mathbb{Z}^g} \exp\Big[2\pi \boldsymbol{i}\Big\{\frac{1}{2}{}^t(n+\delta'){\omega'}^{-1}\omega''(n+\delta') \\
&\qquad\qquad + {}^t(n+\delta')({\omega'}^{-1}\,{}^t u + \delta'')\Big\}\Big],
\end{aligned} \tag{4.4}$$

where

$$c = \frac{1}{\sqrt[8]{D}} \left(\frac{\pi^g}{|\omega'|}\right)^{1/2} \tag{4.5}$$

with D from (3.8). Here the sign of a root of (4.5) is chosen so that the leading terms of $\sigma(u)$ is just the Schur-Weierstrass polynomial that is originally defined in Buchstaber *et al.* [6]). However, we use the modified version from Ônishi [15], pg. 711. The series (4.4) converges because the imaginary part of ${\omega'}^{-1}\omega''$ is positive-definite.

In what follows, for a given $u \in \mathbb{C}^g$, we denote by u' and u'' the unique elements in $\mathbb{R}^g$ such that

$$u = u'\omega' + u''\omega''. \tag{4.6}$$

Then for u, $v \in \mathbb{C}^g$, and ℓ $(= \ell'\omega' + \ell''\omega'') \in \Lambda$, we define

$$\begin{aligned}
L(u, v) &:= u(\eta'\,{}^t v' + \eta''\,{}^t v''), \\
\chi(\ell) &:= \exp[\pi\boldsymbol{i}\big(2(\ell'\delta'' - \ell''\delta') + \ell'\,{}^t\ell''\big)]\ (\in \{1, -1\}).
\end{aligned} \tag{4.7}$$

In this situation, the most important properties of $\sigma(u; M)$ are as follows:

Lemma 4.8. *The function $\sigma(u)$ is an entire function which is independent of choice of the paths α_j, β_j of* (4.1)*. For all $u \in \mathbb{C}^g$, $\ell \in \Lambda$ and we have*

$$\sigma(u+\ell) = \chi(\ell)\sigma(u) \exp L\Big(u + \frac{1}{2}\ell, \ell\Big), \tag{4.9}$$

$$u \mapsto \sigma(u) \text{ has zeroes of order 1 along } \Theta^{[g-1]}, \tag{4.10}$$

$$\sigma(u) = 0 \iff u \in \Theta^{[g-1]}, \tag{4.11}$$

where $\Theta^{[g-1]}$ is as defined following (3.6)*.*

Proof. These are essentially classical results, and can be proved as in [8], Lemma 4.1. So we omit the proof. □

Lemma 4.12. *The coefficients in the expansion of the function $\sigma(u)$ at the origin are polynomials of μ_js in* (2.1) *over the rationals $\mathbb{Q}$.*

Proof. See Nakayashiki [14]. □

Lemma 4.13. *Let χ and L be defined as above. The space of entire functions $\varphi(u)$ on $\mathbb{C}^g$ satisfying*

$$\varphi(u+\ell) = \chi(\ell)\varphi(u) \exp L\Big(u + \frac{1}{2}\ell, \ell\Big)$$

is 1*-dimensional.*

Proof. This is shown by the fact that the Pfaffian of the Riemann form attached to $L(\ ,\ )$ is 1 (see Lang [12], p.93, Th.3.1). □

Definition 4.14. *We introduce a* weight *by letting the weight of μ_j to be $-j$, that of u_j to be $2(g-j)+1$, that of x to be -2, and that of y to be $-2g-1$.*

We can easily show that any formula on J is a sum of terms homogeneous in this weight. In many cases, the computation will be easier if the weight is taken into consideration. We can further simplify by subdividing the calculations according to the separate weights of the μ_i and the u_j terms.

4.2 Complex multiplication of $\sigma(u)$

Lemma 4.15. *Let C be a $(2, a[m])$- or $(2, a[m]+1)$-curve. Let $\sigma(u)$ be the sigma function associated with C as above. Let*

$$w = \begin{cases} ((am)^2-1)/8, & \text{if } am \text{ is odd,} \\ ((am+1)^2-1)/8, & \text{if } am \text{ is even.} \end{cases}$$

Let $\zeta = \exp(2\pi \boldsymbol{i}/(2a))$. By the map 3.6, the action (2.2) on the curve C induce naturally an action of the group W_{2a} of $2a$-th roots of 1 on the space $\mathbb{C}^g$ where u varies. We write this action explicitly for each cases below. Then we see that

$$\sigma([-\zeta]u) = (-\zeta)^w \sigma(u).$$

Proof. Let Λ be the lattice in $\mathbb{C}^g$ as above. Then we have

$$[-\zeta]\Lambda = \Lambda.$$

By (4.13), there is a constant K such that

$$\sigma([-\zeta]u) = K\sigma(u).$$

Because $[-\zeta]^{2a}$ is the identity on $\mathbb{C}^g$, we see

$$K^{2a} = 1.$$

By looking at the leading terms of $\sigma(u)$, we have

$$K = (-\zeta)^w,$$

as desired. $\square$

5 $\wp$-functions

Using the sigma functions defined in the previous section, we let

$$\wp_{jk}(u) = -\frac{\partial^2}{\partial u_j \partial u_k} \log \sigma(u), \quad \wp_{jk\ell}(u) = \frac{\partial}{\partial u_\ell} \wp_{jk}(u), \quad \text{etc.} \tag{5.1}$$

Then by (4.9), these functions are periodic with respect to the Λ of (3.5). If the genus of C is $g = 1$, then, as usual, we write more classically $\wp_{11}(u) = \wp(u)$ and $\wp_{111}(u) = \wp'(u)$.

6 Genus One

6.1 Generalities

For completeness we start off with C be the general elliptic curve defined by

$$y^2 + (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6.$$

Then the $\wp(u)$ defined by (5.1) satisfies

$$\wp'(u) = 2y + \mu_1 x + \mu_3, \quad \wp(u) = x$$

when

$$u = \int_\infty^{(x,y)} \frac{dx}{2y + \mu_1 x + \mu_3},$$

and the $\sigma(u)$, $\wp(u)$ satisfy (1.5) in the Introduction.

6.2 Equianharmonic case

We now specialize C to the curve $y^2 + \mu_3 y = x^3 + \mu_6$. Then we have

$$(\wp')^2 = 4\wp^3 + 4({\mu_3}^2 + \mu_6). \tag{6.1}$$

As usual, by putting $g_3 = -4({\mu_3}^2 + \mu_6)$, we rewrite (6.1) as

$$(\wp')^2 = 4\wp^3 - g_3. \tag{6.2}$$

This is usually called the *equianharmonic* case (see Abramowitz & Stegun [1]). Let $\zeta = \exp(2\pi \boldsymbol{i}/3)$. Then $\zeta^2 = -\zeta - 1$, and

$$\sigma(\zeta u) = \zeta\sigma(u), \quad \wp(\zeta u) = \zeta\wp(u), \quad \wp'(\zeta u) = \wp'(u), \tag{6.3}$$

by (4.15).

The main results for the equianharmonic case are two novel addition formulae, one for two variables and one for three variables:

Proposition 6.4.

$$-\frac{\sigma(u \pm v)\sigma(u \pm \zeta v)\sigma(u \pm \zeta^2 v)}{\sigma(u)^3\sigma(v)^3} = \pm\frac{1}{2}\left(\wp'(u) \pm \wp'(v)\right). \tag{6.5}$$

$$\frac{\sigma(u+v+w)\sigma(u+\zeta v+\zeta^2 w)\sigma(u+\zeta^2 v+\zeta w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3} = \tag{6.6}$$

$$\frac{1}{4}(\wp'(u)\wp'(v) + \wp'(u)\wp'(w) + \wp'(v)\wp'(w)) - \frac{3}{4}(4\wp(u)\wp(v)\wp(w) - g_3).$$

Proof. We give two proofs of these results, the first based on straightforward manipulations of (1.5) and (1.6), and the second based on a pole argument. As we use both techniques in the genus 2 case, we give some detail here for completeness, and to aid understanding.

First Proof. In (1.5), put $v = \zeta u$ and use (6.3) to get

$$\frac{\sigma((1-\zeta)u)}{\sigma(u)^3} = (1-\zeta)\wp(u).$$

Next put $w = \zeta u$ in (1.6) and use the above result, the fact that σ is an odd function of its argument, and (6.3) to give (6.6).

Now consider (6.6). Firstly, we make use of (1.5) by taking (u, v) as (v, w), $(\zeta v, \zeta^2 w)$, $(\zeta^2 v, \zeta w)$ in turn. Multiplying all three versions together, we get

$$\begin{aligned}&\frac{\prod_{j=0}^{j=2}\sigma(u-\zeta^{2j}w)\sigma(\zeta^j v-\zeta^{2j}w)\sigma(u-\zeta^j v)\sigma(u+\zeta^j v+\zeta^{2j}w)}{\prod_{j=0}^{j=2}\sigma(u)\sigma(\zeta^j v)^3\sigma(\zeta^{2j}w)^3}\\ &= -\frac{1}{8}\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(v) & \wp'(v) \\ 1 & \wp(w) & \wp'(w)\end{vmatrix}\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(\zeta v) & \wp'(\zeta v) \\ 1 & \wp(\zeta^2 w) & \wp'(\zeta^2 w)\end{vmatrix}\begin{vmatrix} 1 & \wp(u) & \wp'(u) \\ 1 & \wp(\zeta^2 v) & \wp'(\zeta^2 v) \\ 1 & \wp(\zeta w) & \wp'(\zeta w)\end{vmatrix}\end{aligned} \tag{6.7}$$

Now note the denominator of the l.h.s. simplifies using (6.3) to

$$\sigma(u)^9\sigma(v)^9\sigma(w)^9.$$

Consider now the r.h.s. Multiply this out, simplify using (6.3), then replace all occurrences of $\wp(\cdot)^3$ with $\frac{1}{4}(\wp'(\cdot)^2+g_3)$. Then factor (Maple is useful for this calculation!). The result is

$$\begin{aligned}&-\frac{1}{32}(\wp'(u)-\wp'(w))(\wp'(v)-\wp'(w))(\wp'(v)-\wp'(u))\times\\&\qquad\times(\wp'(v)\wp'(u)+\wp'(v)\wp'(w)+\wp'(w)\wp'(u)+3g_3-12\wp(u)\wp(v)\wp(w))\end{aligned}$$

Now apply (6.5) with the minus sign and with $(u,v)=(u,v),(u,w),(v,w)$ in turn to the numerator of the l.h.s. of (6.7). Finally, cancelling common factors, we have (6.6).

Second Proof. Both sides of (6.5) are elliptic functions, by (4.9). Fixing u and regarding both sides of (6.5) as a function of v, we see both sides have the same poles and zeroes with the same order at

$$v=0 \text{ of order } -3, \quad u \text{ of order } 1, \quad \zeta u \text{ of order } 1, \quad \zeta^2 u \text{ of order } 1,$$

and with no poles or zeroes elsewhere, because of (6.3). Hence the two sides coincide up to a non-zero multiplicative constant. Looking at the coefficients of Laurent expansion with respect to v, we see the two sides are equal.

Now consider (6.6). Recall that, in this case, J is isomorphic to C, and the space of functions on J having a pole only at 0 of order at most n is given by

$$\Gamma(J,\mathcal{O}(n\cdot\circ))=\begin{cases}\mathbb{C} & \text{if } n=0 \text{ or } 1,\\ \mathbb{C}\oplus\mathbb{C}\,\wp(u) & \text{if } n=2,\\ \Gamma(J,\mathcal{O}((n-2)\cdot\circ))\oplus\mathbb{C}\,\wp^{(n-1)}(u) & \text{if } n\geq 3,\end{cases}$$

where we denote $\Theta^{[0]}$ by $\circ$, that is the origin of J. The function $\sigma(u)$ is expanded as

$$\sigma(u)=u-\tfrac{1}{120}\,({\mu_3}^2+\mu_6)\,u^7+O(u^{13}).$$

The left hand side of (6.6) is invariant under $u\leftrightarrow\zeta u$, $v\leftrightarrow\zeta v$, $w\leftrightarrow\zeta w$, and all exchanges of u, v, and w. It is an even function under $u\leftrightarrow -u$, $v\leftrightarrow -v$, $w\leftrightarrow -w$ simultaneously. Moreover, it is of homogeneous weight -6. Hence, it must be of the form

$$a\left(\wp'(u)\wp'(v)+\wp'(u)\wp'(w)+\wp'(v)\wp'(w)\right)+b\,\wp(u)\wp(v)\wp(w)+c\,\mu_6,$$

where a, b, c are constants independent of g_3. Then, by using first few terms of the power series expansion with respect to u or v, and by balancing the two sides, we determine these coefficients to obtain (6.5). □

Remark 6.8. *In the "rational" case, $\mu_3=\mu_6=0$, $\sigma(u)=u$, $\wp(u)=1/u^2$, the formula* (6.6) *becomes the well-known identity*

$$(a+b+c)(a+\zeta b+\zeta^2 c)(a+\zeta^2 b+\zeta c)=a^3+b^3+c^3-3abc.$$

Remark 6.9. *Formula (6.6) turns up as a special case in the study of exceptional completely decomposable quasi-linear (CDQL) webs globally defined on compact complex surfaces [16].*

6.3 Lemniscate case (the $(2, 2[1]+1)$-curve)

For the curve $y^2 = x^3 + \mu_4 x$, we have

$$\left.\begin{aligned} \wp'(u) &= 2y \\ \wp(u) &= x \end{aligned}\right\} \text{ if } u = \int_\infty^{(x,y)} \frac{dx}{2y+\mu_3}, \tag{6.10}$$

and

$$(\wp')^2 = 4\wp^3 + 4\mu_4\wp. \tag{6.11}$$

As usual by putting $\mu_4 = -g_2/4$, we rewrite (6.11) as

$$(\wp')^2 = 4\wp^3 - g_2\wp. \tag{6.12}$$

This is usually called the *lemniscate* case (see Abramowitz & Stegun [1]). By (4.15), we see that, for the $\wp$ satisfying (6.12),

$$\sigma(\boldsymbol{i}u) = \boldsymbol{i}\sigma(u), \quad \wp(\boldsymbol{i}u) = -\wp(u), \quad \wp'(\boldsymbol{i}u) = \boldsymbol{i}\wp'(u). \tag{6.13}$$

In this case, we have from (1.5) with $v \to iv$ and (6.13) that

$$-\frac{\sigma(u+iv)\sigma(u-iv)}{\sigma(u)^2\sigma(v)^2} = \wp(u) + \wp(v). \tag{6.14}$$

Generalizing (1.5) and this, the main results for the lemniscate case are the following addition formulae:

Proposition 6.15.

$$\begin{aligned} &\frac{\sigma(u+v+w)\sigma(u+v-w)\sigma(u-v+w)\sigma(u-v-w)}{\sigma(u)^4\sigma(v)^4\sigma(w)^4} \\ &\quad= \frac{1}{16}g_2^2 + \frac{1}{2}g_2\left(\wp(v)\wp(w) + \wp(u)\wp(w) + \wp(u)\wp(v)\right) \\ &\quad+ \wp(u)^2\wp(v)^2 + \wp(u)^2\wp(w)^2 + \wp(w)^2\wp(v)^2 \\ &\quad- 2\wp(u)\wp(v)\wp(w)(\wp(u) + \wp(v) + \wp(w)) \equiv E_0(u,v,w). \end{aligned} \tag{6.16}$$

$$\begin{aligned} &\frac{\sigma(u+iv+w)\sigma(u+iv-w)\sigma(u-iv+w)\sigma(u-iv-w)}{\sigma(u)^4\sigma(v)^4\sigma(w)^4} \\ &\quad= \frac{1}{16}g_2^2 + \frac{1}{2}g_2(\wp(u)\wp(w) - \wp(v)\wp(w) - \wp(u)\wp(v)) \\ &\quad+ \wp(u)^2\wp(v)^2 + \wp(u)^2\wp(w)^2 + \wp(w)^2\wp(v)^2 \\ &\quad+ 2\wp(v)\wp(u)\wp(w)(\wp(w) - \wp(v) + \wp(u)) \equiv E_1(u,w;v). \end{aligned} \tag{6.17}$$

By symmetry we have two further formulae under the transformations $u \to iu$ and $w \to iw$, and finally we have the 16-term formula

$$\frac{\prod_{n,m=0,1,2,3}\sigma(u+i^n v+i^m w)}{\sigma(u)^{16}\sigma(v)^{16}\sigma(w)^{16}} = E_0(u,v,w)E_1(u,v;w)E_1(u,w;v)E_1(v,w;u).$$

Proof. The 4-term formulae are constructed from products of the relation (1.6) in the same way as in §6.2. Similar relations for other permutations of terms can also be constructed. $\square$

Remark 6.18. *The addition formulae in this section can be proved by another method, as in §7.3 below.*

7 Genus Two

In this section we treat curves of genus two. So, $am = 4$ or 5.

7.1 Basis of spaces of Abelian functions

Using the functions in (5.1), we denote

$$\begin{aligned} \Delta &= \det[\wp_{ij}] = \wp_{11}\wp_{22} - {\wp_{12}}^2, \\ \Delta_j &= \frac{\partial}{\partial u_j}\Delta, \quad \Delta_{ij} = \frac{\partial^2}{\partial u_j \partial u_i}\Delta, \quad \text{etc.} \end{aligned} \tag{7.1}$$

Lemma 7.2. *Let $n \geq 2$ be an integer. The space $\Gamma(J, \mathcal{O}(n\Theta^{[1]})$ of the functions having no pole outside $\Theta^{[1]}$ and at most of order n on $\Theta^{[1]}$ is given recursively by*

$$\Gamma(J, \mathcal{O}(2\Theta^{[1]})) = \mathbb{C}\,1 \oplus \mathbb{C}\wp_{11} \oplus \mathbb{C}\wp_{12} \oplus \mathbb{C}\wp_{22},$$
$$\Gamma(J, \mathcal{O}\big((n+1)\Theta^{[1]}\big)) = \tfrac{\partial}{\partial u_1}\Gamma(J, \mathcal{O}(n\Theta^{[1]})) \cup \tfrac{\partial}{\partial u_2}\Gamma(J, \mathcal{O}(n\Theta^{[1]})).$$

In particular,

$$\begin{aligned} \Gamma(J, \mathcal{O}(3\Theta^{[1]})) &= \Gamma(J, \mathcal{O}(2\Theta^{[2]})) \oplus \mathbb{C}\wp_{111} \oplus \mathbb{C}\wp_{112} \oplus \mathbb{C}\wp_{122} \oplus \mathbb{C}\wp_{222} \oplus \mathbb{C}\,\Delta, \\ \Gamma(J, \mathcal{O}(4\Theta^{[1]})) &= \Gamma(J, \mathcal{O}(3\Theta^{[1]})) \oplus \mathbb{C}\wp_{1111} \oplus \mathbb{C}\wp_{1112} \oplus \mathbb{C}\wp_{1122} \oplus \mathbb{C}\wp_{1222} \oplus \mathbb{C}\wp_{2222} \\ &\quad \oplus \mathbb{C}\Delta_1 \oplus \mathbb{C}\Delta_2, \\ \Gamma(J, \mathcal{O}(5\Theta^{[1]})) &= \Gamma(J, \mathcal{O}(4\Theta^{[1]})) \oplus \mathbb{C}\wp_{11111} \oplus \mathbb{C}\wp_{11112} \oplus \mathbb{C}\wp_{11122} \oplus \mathbb{C}\wp_{11222} \\ &\quad \oplus \mathbb{C}\wp_{12222} \oplus \mathbb{C}\wp_{22222} \oplus \mathbb{C}\Delta_{11} \oplus \mathbb{C}\Delta_{12} \oplus \mathbb{C}\Delta_{22}. \end{aligned}$$

For the convenience to the reader, we list their weight below:

function	$\wp_{11}$	$\wp_{12}$	$\wp_{22}$	$\wp_{111}$	$\wp_{112}$	$\wp_{122}$	$\wp_{222}$	Δ	Δ_1	Δ_2	$\wp_{1111}$
weight	-6	-4	-2	-9	-7	-5	-3	-8	-11	-9	-12

$\wp_{1112}$	$\wp_{1122}$	$\wp_{1222}$	$\wp_{2222}$	$\wp_{11111}$	$\wp_{11112}$	$\wp_{11122}$	$\wp_{11222}$	$\wp_{12222}$	$\wp_{22222}$
-10	-8	-6	-4	-15	-13	-11	-9	-7	-5

Proof. This is shown in Cho & Nakayashiki [7], see especially the example for $g = 2$ in Section 9 of that paper. $\square$

7.2 Equipentamic case

We propose the name "Equipentamic" for the $(2, 5[1])$-curve

$$f(x, y) = y^2 - (x^5 + \mu_{10}).$$

In this case we have

$$\begin{aligned}
&\wp_{11}([-\zeta]u) = \zeta^3\wp_{11}(u),\ \ \wp_{12}([-\zeta]u) = \zeta^2\wp_{12}(u),\ \ \wp_{22}([-\zeta]u) = \zeta\wp_{22}(u),\\
&\wp_{111}([-\zeta]u) = -\zeta^2\wp_{111}(u),\ \ \cdots
\end{aligned}$$

for $\zeta = \zeta_5 = \exp(2\pi\boldsymbol{i}/5)$, because of 4.15.

Proposition 7.3. *We have*

$$\begin{aligned}
&\frac{\sigma(u+v)\sigma(u+[\zeta]v)\sigma(u+[\zeta^2]v)\sigma(u+[\zeta^3]v)\sigma(u+[\zeta^4]v)}{\sigma(u)^5\sigma(v)^5}\\
&= \tfrac{5}{18}[\wp_{122}(u)\wp_{1112}(v)+\wp(v)\wp_{1112}(u)] - \tfrac{5}{144}[\wp_{122}(u)\Delta_{22}(v)+\wp_{122}(v)\Delta_{22}(u)]\\
&\quad - \tfrac{1}{144}[\wp_{1112}(u)\wp_{22222}(v)+\wp_{1112}(v)\wp_{22222}(u)] - \tfrac{1}{24}[\wp_{11111}(u)+\wp_{11111}(v)]\\
&\quad - \tfrac{1}{576}[\Delta_{22}(u)\wp_{22222}(v)+\Delta_{22}(v)\wp_{22222}(u)] + \tfrac{1}{24}\mu_{10}[\wp_{22222}(u)+\wp_{22222}(v)],
\end{aligned}\tag{7.4}$$

where $\zeta = \exp(2\pi\boldsymbol{i}/5)$, *and* $[\zeta]v = [\zeta](v_1, v_2) = (\zeta v_1, \zeta^2 v_2)$. *Alternatively, the r.h.s. of the relation above can be written as*

$$\begin{aligned}
&= \tfrac{1}{4}\wp_{22}(u)\wp_{222}(u)\big(\wp_{22}(v)\wp_{12}(v)^2 - \wp_{11}(v)\wp_{22}(v)^2 - 4\wp_{12}(v)\wp_{11}(v)\big)\\
&\quad + \tfrac{1}{2}\wp_{122}(u)\big(\wp_{12}(v)\wp_{11}(v) + \wp_{22}(v)\wp_{12}(v)^2 - \wp_{11}(v)\wp_{22}(v)^2\big)\\
&\quad - \tfrac{1}{2}\wp_{11}(u)\wp_{111}(u) + \mu_{10}\wp_{22}(u)\wp_{222}(u) + (u \Leftrightarrow v).
\end{aligned}\tag{7.5}$$

Proof. The left hand side is an Abelian function of weight -15 as a function of u (resp. v) because of (4.9), and has poles only along $\Theta^{[1]}$ of order 5 by (4.10), (4.11). It is also invariant under $u \leftrightarrow [\zeta]u$, $v \leftrightarrow [\zeta]v$, and $u \leftrightarrow v$. According to Lemma 4.12, it must be a linear combination of homogeneous weight -15 of terms of the form

$$c_j\,\big(X_j(u)Y_j(v) + X_j(v)Y_j(u)\big),\tag{7.6}$$

where $X_j(u)$ and $Y_j(u)$ are members of the list just below the Lemma 7.2, with coefficients c_j being polynomial of μ_{10} over the rationals. So, there are 6 possible terms of (7.6), namely those in appearing in the right hand side of (7.4). Its coefficients follows from expanding both sides in power series in u and v with first several terms after multiplying $\sigma(u)^5\sigma(v)^5$ to both sides, by computer calculation using Maple. To prove (7.4), we use the known expansions of the 4-index $\wp_{ijkl}$ relations in this case

$$\begin{aligned}
&\wp_{2222} = 6\wp_{22}^2 + 4\wp_{12}, &&\wp_{1222} = 6\wp_{22}\wp_{12} - 2\wp_{11},\\
&\wp_{1122} = 4\wp_{12}^2 + 2\wp_{11}\wp_{22}, &&\wp_{1112} = 6\wp_{12}\wp_{11} - 4\mu_{10}\\
&\wp_{1111} = 6\wp_{11}^2 - 12\wp_{22}\mu_{10},
\end{aligned}$$

together with the derivatives of these equations with respect to the u_i. In addition we use the known quadratic 3-index relations

$$\begin{aligned}\wp_{222}^2 &= 4\wp_{22}^3 + 4\wp_{11} + 4\wp_{22}\wp_{12},\\ \wp_{122}\wp_{222} &= -2\wp_{22}\wp_{11} + 4\wp_{22}^2\wp_{12} + 2\wp_{12}^2,\\ \ldots &= \ldots\end{aligned}$$

These substitutions lead eventually to (7.5). □

Remark 7.7. *Other types of addition formulae exist, for example, for u, v, $w \in \mathbb{C}^2$,*

$$\frac{\sigma(u+v+w)\sigma(u+[\zeta]v+[\zeta]^2w)\sigma(u+[\zeta]^2v+[\zeta]^4w)\sigma(u+[\zeta]^3v+[\zeta]w)\sigma(u+[\zeta]^4v+[\zeta]^3w)}{\sigma(u)^5\sigma(v)^5\sigma(w)^5} \tag{7.8}$$

is expressed in terms of $\wp$-functions and their derivatives. But it would need a big calculation to get the explicit expression.

7.3 The $(2, 2[2]{+}1)$-curve

Here we treat the $(2, 2[2]{+}1)$-curve C given by

$$f(x,y) = y^2 - (x^5 + \mu_4 x^3 + \mu_8 x). \tag{7.9}$$

The result here is not so interesting because it is essentially a product of two of (1.9) in the Introduction. For completeness we describe the result here in compressed form. This curve C has the automorphism

$$[\boldsymbol{i}] : (x,y) \mapsto (-x, \boldsymbol{i}y), \quad \left(\frac{dx}{2y}, \frac{xdx}{2y}\right) \mapsto \left(\boldsymbol{i}\frac{dx}{2y}, -\boldsymbol{i}\frac{xdx}{2y}\right)$$

We see that

$$\begin{aligned} &[\boldsymbol{i}]^2 = [-1] : (x,y) \mapsto (x,-y),\\ &[\boldsymbol{i}] : (u_1,u_2) \mapsto (\boldsymbol{i}u_1, -\boldsymbol{i}u_2), \quad [\boldsymbol{i}]^2(u_1,u_2) \mapsto (-u_1,-u_2),\\ &\wp_{11}([\boldsymbol{i}]u) = -\wp(u), \quad \wp_{12}([\boldsymbol{i}]u) = \wp_{12}(u),\\ &\wp_{22}([\boldsymbol{i}]u) = -\wp(u), \quad \wp_{111}([\boldsymbol{i}]u) = \boldsymbol{i}\wp_{12}(u), \quad \text{etc..}\end{aligned} \tag{7.10}$$

We trivially have the following formula.

$$\begin{aligned}&\frac{\sigma(u+v)\sigma(u+[\boldsymbol{i}]v)\sigma(u+[\boldsymbol{i}]^2v)\sigma(u+[\boldsymbol{i}]^3v)}{\sigma(u)^4\sigma(v)^4}\\ &\qquad = (\wp_{11}(u) - \wp_{11}(v) + \wp_{12}(u)\wp_{22}(v) - \wp_{22}(u)\wp_{12}(v))\\ &\qquad\quad \times (\wp_{11}(u) + \wp_{11}(v) - \wp_{12}(u)\wp_{22}(v) - \wp_{12}(v)\wp_{22}(u)).\end{aligned} \tag{7.11}$$

This is not new because $[\boldsymbol{i}]^2$ is no other than the standard involution $u \to -u$. We remark further on this in the final section.

8 Higher genus and non-hyperelliptic curves

These new results described above for genus one and two were inspired by results for the trigonal genus three case [8]. In that paper we derived a three-term two-variable addition formula which generalises (6.5) to the purely trigonal case of genus three

$$f(x,y) = y^3 - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}). \tag{8.1}$$

We have also recently proved the existence of a similar three-term two-variable addition formula for the purely trigonal case of genus four [4]

$$f(x,y) = y^3 - (x^5 + \mu_3 x^4 + \mu_6 x^3 + \cdots + \mu_{15}). \tag{8.2}$$

In addition we showed in this paper the existence of three-term three-variable addition formula for (8.1) and (8.2) which generalise (6.6).

8.1 The three-term three-variable addition formula for the (3,4) purely trigonal curve

The formula in this case generalises the three-variable formula given in (6.6) for the genus 1 case. This type of formula is expected to be quite complicated, as the family of members in the corresponding natural basis in each case is large.

For example in the case of (8.1), we have an addition formula of the type

$$\begin{aligned} &\frac{\sigma(u+v+w)\sigma(u+[\zeta]v+[\zeta^2]w)\sigma(u+[\zeta^2]v+[\zeta]w)}{\sigma(u)^3\sigma(v)^3\sigma(w)^3} \\ &\qquad = \sum_{i=1}^{27}\sum_{j=1}^{27}\sum_{k=1}^{27} c_{ijk} U_i(u) V_j(v) W_k(w). \end{aligned} \tag{8.3}$$

where the functions U_i, V_j W_k are all basis functions for the space $\Gamma(J, \mathcal{O}(3\Theta^{[2]}))$. These are enumerated in [8]. We can write the r.h.s. as

$$C_{30} + C_{27} + \dots C_0,$$

where C_n has weight $-n$ in u, v, w combined, and weight $n - 30$ in the $\lambda_i, i = 3, \dots, 0$. Both the l.h.s. and the r.h.s. are symmetric under all permutations in (u, v, w). So far only the terms from C_{30} up to C_{18} have been calculated, and a full description of this formula will be given elsewhere. To illustrate the complexity we give here the formula for C_{30}

$$
\begin{aligned}
C_{30} = {} & \tfrac{1}{6}\wp_{13}(u)\partial_3 Q_{1333}(v)\wp_{111}(w) + \tfrac{1}{48}\partial_3 Q_{1333}(u)\partial_3 Q_{1333}(v)\wp^{[22]}(w) \\
& - \tfrac{3}{2}\wp_{12}(u)\wp_{11}(v)\wp^{[23]}(w) - \wp_{13}(u)\wp^{[22]}(v)\wp^{[22]}(w) \\
& + \tfrac{1}{8}\wp^{[12]}(u)\partial_3 Q_{1333}(v)\wp_{112}(w) - \tfrac{1}{2}\wp_{11}(u)\wp_{11}(v)\wp_{11}(w) \\
& - \tfrac{3}{16}\wp_{222}(u)\wp^{[22]}(v)\wp_{112}(w) - \tfrac{3}{8}\wp_{22}(u)\wp^{[23]}(v)\wp^{[23]}(w) \\
& - \tfrac{1}{8}\wp^{[11]}(u)\wp^{[22]}(v)\wp^{[22]}(w) + \tfrac{3}{16}\wp_{122}(u)\wp_{122}(v)\wp^{[22]}(w) \\
& - \tfrac{1}{8}\wp^{[13]}(u)\wp^{[13]}(v)\wp^{[13]}(w) + \tfrac{3}{8}\wp_{123}(u)\wp_{123}(v)\wp^{[33]}(w) \\
& - \tfrac{1}{8}Q_{1333}(u)Q_{1333}(v)\wp^{[33]}(w) - \tfrac{3}{8}\wp_{33}(u)\wp^{[33]}(v)\wp^{[33]}(w) \\
& - \tfrac{1}{8}\wp_{133}(u)\partial_2 Q_{1333}(v)\wp^{[23]}(w) + \tfrac{1}{4}\wp_{133}(u)\wp_{11}(v)\partial_1 Q_{1333}(w) \\
& - \tfrac{1}{8}\wp_{133}(u)\wp^{[13]}(v)\partial_1 Q_{1333}(w) - \tfrac{3}{8}\wp_{223}(u)\wp_{113}(v)\wp^{[33]}(w) \\
& + \tfrac{1}{4}\wp^{[12]}(u)\wp^{[12]}(v)\wp^{[22]}(w) - \tfrac{1}{4}P(1,3)\wp_{122}(v)\wp_{111}(w) \\
& + \tfrac{1}{8}\wp_{111}(u)\wp_{111}(v) + \tfrac{3}{8}Q_{1333}(u)\wp_{113}(v)\wp_{113}(w) \\
& + \text{all permutations of} \quad (u,v,w).
\end{aligned}
$$

8.2 Other formulae for higher genus curves

There are a number of other addition formulae waiting to be computed in explicit form for special cases of $g > 2$ hyperelliptic curves and for other special cases of trigonal and curves with higher gonal numbers.

For instance, let

$$
f(x,y) = \begin{cases} y^3 - (x^{am} + \mu_{3a}x^{a(m-1)} + \mu_{6a}x^{a(m-2)} + \cdots \\ \qquad\qquad + \mu_{3a(m-1)}x^{a} + \mu_{3am}), \qquad \text{if } \gcd(am,3)=1, \\ y^3 - (x^{am+1} + \mu_{3a}x^{a(m-1)+1} + \mu_{6a}x^{a(m-2)+1} + \cdots \\ \qquad\qquad + \mu_{3a(m-1)}x^{a+1} + \mu_{3am}x), \quad \text{if } \gcd(am,3)=3; \end{cases}
$$

and C be the curve defined by $f(x,y)=0$. This should be called $(3,a[m])$-curve or $(3,a[m]+1)$-curve, respectively. Let $\zeta = \exp(2\pi\boldsymbol{i}/(3a))$. Then C has automorphisms

$$
[\zeta] : \begin{cases} (x,y) \mapsto (\zeta^3 x, \zeta^a y), & \text{if } \gcd(am,3)=1, \\ (x,y) \mapsto (\zeta^3 x, -\zeta^a y), & \text{if } \gcd(am,3)=3. \end{cases} \tag{8.4}
$$

Namely, it is acted on by W_{3a}, the group of $3a$-th roots of 1. Associated with such a curve C, we will have various multi-term addition formulae.

We shall give a remark by an explicit example. For $(3,4[1])$-curve $y^3 = x^4 + \mu_{12}$ and the automorphism

$$
[\boldsymbol{i}] : (x,y) \mapsto (-\boldsymbol{i}x, y), \tag{8.5}
$$

the action $[\boldsymbol{i}]^2$ is different from standard involution $u \to -u$ on the variable space of associated Abelian functions. The formula that expresses

$$\frac{\sigma(u+v)\sigma(u+[\boldsymbol{i}]v)\sigma(u+[\boldsymbol{i}]^2 v)\sigma(u+[\boldsymbol{i}]^3 v)}{\sigma(u)^4\sigma(v)^4} \tag{8.6}$$

in terms of $\wp$-functions seems to be interesting.

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